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line, all of the four angles formed are right angles and the lines may be said to be perpendicular.

These are the familiar theorems concerning the erection of perpendiculars. The proofs will be left to the reader.

REPRESENTATION OF REAL AND IMAGINARY LOCI IN THE SAME PLANE.

By PROFESSOR G. W. GREENWOOD. McKendree College.

In elementary analytical geometry, where only the real parts of the locii are represented, it is difficult at times to give satisfactory definitions. Take for example the polar of a point with respect to a conic. If we define it as the locus of the intersection of tangents at points where the conic is cut by any secant through the point, we get when the point is without the conic a portion of the polar upon which no tangents meet; if we define it as a chord of contact, our definition appears strained when the point is within the conic.

I have been using, with some success, a method which I have not seen in any text book, and which is in brief as follows:

Consider the circle $x^2 + y^2 = a^2$. Taking value of x greater than a , numerically, we get $y = \pm ib$, say, where b is real. Plot the points obtained by taking the real factors in the values of y and we get the hyperbola

$$x^2 - y^2 = a^2,$$

which we may draw in red ink. This will contain all imaginary points of the circle of the form (c, ib) .

Now either definition of the polar of a point on $y=0$ will apply; if the point is outside the circle, it is within the hyperbola and its polar is the locus of the intersection of real tangents to the circle or imaginary tangents (*i. e.* those to the hyperbola) at points where the locus is cut by secants through the point. When the point is within the circle, the chord of contact joins the points of tangency of the imaginary tangents.

The radical axis of two circles appears in all cases as the line through two common points, real or imaginary.

Two circles are seen to intersect in two infinite imaginary points, and two concentric circles are said to have double contact at infinity, the tangent there being the circular lines.

The limiting forms of the circle and its hyperbola illustrate graphically the dual interpretation of the equation

$$x^2 + y^2 = 0,$$

as a point circle and a pair of lines.

If we treat the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

in like manner we see, among other things, that the tangents from a focus are perpendicular and that their chord of contact is the corresponding directrix.

Reciprocally, the ellipse supplements the hyperbola in picturing the imaginary part.

TRIANGULAR RESIDUES.

By ORLANDO S. STETSON, Syracuse University.

The summation of arithmetical progressions which commence with unity and whose differences are 1, 2, 3, etc., leads to the theory of *polygonal numbers*.

Employing the difference 1, we obtain the *triangular numbers*. The π th one is $\frac{1}{2}\pi(\pi+1)$. If ρ be any prime number and the series of the first $\rho-2$ triangular numbers be formed, the least positive remainders $(\text{mod } \rho)$ of the first $\frac{1}{2}(\rho-1)$ of these numbers will be all different. Indeed, if two of the remainders were equal for values π_1 and π_2 of π , we should have

$$(\pi_1 - \pi_2)(\pi_1 + \pi_2 + 1) \equiv 0 \pmod{\rho},$$

which is impossible. The remainder of the middle $[\frac{1}{2}(\rho-1)]$ th term $\frac{1}{8}(\rho^2-1)$ occurs once only, while the remainders of the remaining terms repeat themselves in reverse order.

The $\frac{1}{2}(\rho-1)$ different remainders are called *triangular residues* and the remaining numbers of the series 1, 2, ..., $\rho-1$, are called *triangular non-residues*.

Let λ be the least positive remainder of the middle term $\frac{1}{8}(\rho^2-1)$. Then

$$(1) \quad 8\lambda + 1 \equiv 0 \pmod{\rho}.$$

Theorem I.* If ρ be any prime number excepting 3, the sum of the triangular residues $\equiv \mu \pmod{\rho}$, where $2\mu \equiv \lambda \pmod{\rho}$.

If $f(\pi)$ denote the sum of $1+3+5+\dots+\frac{\pi(\pi+1)}{2}$, then $S_t \equiv f\left(\frac{\rho-1}{2}\right)$, where S_t denotes the sum of all the triangular residues. Now,

$$2f(\pi) = 2 \sum_{h=1}^{\pi} \frac{1}{2}h(h+1) = \sum_{h=1}^{\pi} (h^2 + h).$$

Summing the series and multiplying both sides of the congruence by 3, we have

*Proved first by M. Stern, *Journal für Mathematik*, Bd. 69.